

Exceptional points and unitary evolution of the physical solutions

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Abstract. An example of exceptional points in the continuous spectrum of a real, pseudo-Hermitian Hamiltonian of von Neumann-Wigner type is presented and discussed. Remarkably, these exceptional points are associated with a double pole in the normalization factor of the Jost eigenfunctions normalized to unit flux at infinity. At the exceptional points, the two unnormalized Jost eigenfunctions are no longer linearly independent but coalesce to give rise to two Jordan cycles of generalized bound state eigenfunctions embedded in the continuum and a Jordan block representation of the Hamiltonian. The regular scattering eigenfunction vanishes at the exceptional point and the irregular scattering eigenfunction has a double pole at that point. In consequence, the time evolution of the regular scattering eigenfunction is unitary, while the time evolution of the irregular scattering eigenfunction is pseudounitary. The scattering matrix is a regular analytical function of the wave number k for all k including the exceptional points.

PACS numbers: 02.40.Xx, 03.65.Nk, 03.65.Vf

1. Introduction

Non-Hermitian Hamiltonians are widely used to describe open quantum systems in many fields of science [1–7]. For continuum, PT -symmetric, non-Hermitian Hamiltonians on an infinite line Bender et al. [8, 9] showed that the eigenvalue spectrum is purely real when the strength of the non-Hermiticity is small. Recently, Yogesh N et al. have investigated the signature of PT -symmetry breaking in coupled waveguides [10]. On the other hand, the readings in a measuring device are real numbers which, according to Quantum Mechanics, correspond to the points in the spectrum of an operator that represents an observable. This physical condition is stated in mathematical form by demanding that the spectrum of an operator representing an observable should be real [11]. When the observable is represented by a self-adjoint operator the condition is automatically satisfied. But the reality of the spectrum of an operator does not necessarily mean that the operator is self-adjoint [12, 13]. This non-equivalence of selfadjointness and the reality of the spectrum of an operator was made evident by the discovery and subsequent discussion of a large class of non-Hermitian PT -symmetric Schrödinger Hamiltonian operators with a complex valued potential term but with real energy eigenvalues [9, 14, 15]. Even in the case of a radial Schrödinger Hamiltonian with a real potential term, the reality of the energy spectrum does not necessarily mean that the Hamiltonian is self-adjoint. In many cases, the spectrum of a real non-self-adjoint Hamiltonian is real but differs from the spectrum of self-adjoint ones in two essential features. These are, the possible presence of exceptional points and the possible presence of spectral singularities.

Exceptional points already appear in the finite dimensional case of non-Hermitian Hamiltonian matrices depending on a set of control parameters [16–23], whereas spectral singularities are characteristic features of Hamiltonians having a continuous energy spectrum [7, 23–30]. Hence, they are not possible for finite dimensional operators. Exceptional points in the real continuous spectrum of a Schrödinger Hamiltonian with a real potential have received much less attention than in the finite dimensional case [6, 31, 32]. With the purpose of clarifying the topological nature of the exceptional points in the real, continuous energy spectrum of a quantum system, in the following we will present and discuss an example of exceptional points in a Hamiltonian with a real potential of von Neumann-Wigner type. This potential is generated from the eigenfunctions of a free particle Hamiltonian by means of a four times iterated Darboux transformation when the transformation functions are degenerated in the continuum of eigenfunctions of the free particle Hamiltonian [33, 35].

This paper is organized as follows: In section 2, we generated a Hamiltonian $H[4]$ by means of a four times iterated and completely degenerated Darboux transformation which has two exceptional points in its real and continuous degenerated spectrum. In section 3, we compute the Jost solutions of $H[4]$ normalized to unit probability flux at infinity. In section 4, we show that at $k = \pm q$, the Wronskian of the two unnormalized Jost solutions of $H[4]$ vanishes, this property identifies these points as exceptional points

in the spectrum of the Hamiltonian $H[4]$. Section 5, is devoted to show that at the exceptional points, the Hamiltonian $H[4]$ has a Jordan block matrix representation and a Jordan cycle of generalized eigenfunctions. In section 6, we show that the regular scattering solution vanishes at exceptional points, while the irregular scattering solution has a double pole at $k = \pm q$ and the scattering matrix $S(k)$ is a regular function of k . In section 7, we show that the presence of an exceptional point in the spectrum of $H[4]$ does not alter the unitarity evolution of the regular time dependent wave function. The pseudounitary evolution of the irregular time dependent wave function is examined in section 8. A summary of the main results and conclusions are presented in section 9.

2. The potential $V[4]$

A Hamiltonian that has two exceptional points in its real and continuous spectrum may be generated by means of a four times iterated and completely degenerated Darboux transformation [36,37]. Thus, and according with Crum's generalization of the Darboux theorem [38], the function

$$\psi[4] = \frac{W_2(\phi, \partial_q \phi, \partial_q^2 \phi, \partial_q^3 \phi, e^{\pm ikr})}{W_1(\phi, \partial_q \phi, \partial_q^2 \phi, \partial_q^3 \phi)}, \quad (1)$$

is an eigenfunction of the radial Schrödinger equation with the potential

$$V[4] = V_0 - 2 \frac{d^2}{dr^2} \ln W_1(\phi, \partial_q \phi, \partial_q^2 \phi, \partial_q^3 \phi). \quad (2)$$

In these expressions, $W_1(\phi, \partial_q \phi, \dots, \partial_q^3 \phi)$ is the Wronskian of the transformation function $\phi(q, r)$ and its first derivatives with respect to q , and $W_2(\phi, \partial_q \phi, \dots, \partial_q^3 \phi, e^{\pm ikr})$ is the Wronskian of the transformation function, $\phi(q, r)$, its first three derivatives with respect to q and $e^{\pm ikr}$ which is an eigenfunction of the free particle radial Hamiltonian H_0 with eigenvalue $E = k^2$. The transformation function is

$$\phi(q, r) = \sin(qr + \delta(q)), \quad (3)$$

and $\partial_q \phi$ is shorthand for $\partial \phi / \partial q$. The phase shift $\delta(q)$ in the right side of eq.(3) is a smooth, odd function of the wave number q .

The Wronskian $W_1(\phi, \partial_q \phi, \partial_q^2 \phi, \partial_q^3 \phi) = W_1(q, r)$ is readily computed from (3) and the potential $V[4]$ is obtained from $W_1(q, r)$ and its derivatives with respect to r ,

$$V[4] = -2 \frac{1}{W_1^2(q, r)} \left(W_1''(q, r) W_1(q, r) - W_1'^2(q, r) \right). \quad (4)$$

An explicit expression for $W_1(q, r)$ is the following

$$\begin{aligned} W_1(q, r) = & 16(q\gamma)^4 - 12(q\gamma)^2 + 8(q^3\gamma_2)(q\gamma) - 12(q^2\gamma_1)^2 \\ & + 24[(q^2\gamma_1)(q\gamma) + (q\gamma)^2] \cos 2\theta + 3 \sin^2 2\theta \\ & + [16(q\gamma)^3 - 12q\gamma - 12q^2\gamma_1 - 4q^3\gamma_2] \sin 2\theta, \end{aligned} \quad (5)$$

where

$$\theta(r) = qr + \delta(q), \quad \gamma(r) = \partial_q \theta = r + \gamma_0,$$

$$\begin{aligned}\gamma_0 &= \partial_q \delta(q), & \gamma_1 &= \partial_q^2 \delta(q), \\ \gamma_2 &= \partial_q^3 \delta(q).\end{aligned}\tag{6}$$

For large values of r , the dominant term in the right hand side of eq.(5) is $(q\gamma)^4$ which is positive and grows with r as r^4 . Hence, for large values of r , $W_1(q, r)$ is a positive and increasing function of r . However, if the phase $\delta(q)$ is left unconstrained, $W_1(q, r)$ could take a negative value at the origin of the radial coordinate ($r = 0$), in which case it should vanish for some positive, non-vanishing value of r , giving rise to a singularity of the potential $V[4]$ at that point.

A necessary condition for the validity of the method of the Darboux transformation is that the potential generated should not have any singularities that are not present in the initial potential. In the case under consideration, this condition means that the Wronskian $W_1(q, r)$ should not vanish for any positive value of r . Therefore, to avoid the appearance of singularities in $V[4]$ at finite values of r , we will put the condition

$$W_1(q, 0) > 0.\tag{7}$$

In explicit form, we have

$$\begin{aligned}W_1(q, 0) &= 16\left(q\frac{d\delta(q)}{dq}\right)^4 - 12\left(q\frac{d\delta(q)}{dq}\right)^2 + 8\left(q^3\frac{d^3\delta(q)}{dq^3}\right)\left(q\frac{d\delta(q)}{dq}\right) \\ &\quad - 12\left(q^2\frac{d^2\delta(q)}{dq^2}\right)^2 + 24\left[\left(q^2\frac{d^2\delta(q)}{dq^2}\right)\left(q\frac{d\delta(q)}{dq}\right) + \left(q\frac{d\delta(q)}{dq}\right)^2\right] \\ &\quad \times \cos 2\delta(q) + 3\sin^2 2\delta(q) \\ &\quad + \left[16\left(q\frac{d\delta(q)}{dq}\right)^3 - 12\left(q\frac{d\delta(q)}{dq}\right) - 12\left(q^2\frac{d^2\delta(q)}{dq^2}\right) - 4\left(q^3\frac{d^3\delta(q)}{dq^3}\right)\right] \\ &\quad \times \sin 2\delta(q).\end{aligned}\tag{8}$$

Despite its formidable appearance, this equation may readily be integrated. First, to simplify the notation, we define a new function $t(q)$ as

$$t(q) := \tan \delta(q),\tag{9}$$

then

$$\sin 2\delta(q) = \frac{2t(q)}{1+t^2(q)}\tag{10}$$

and

$$\cos 2\delta(q) = \frac{1-t^2(q)}{1+t^2(q)}.\tag{11}$$

Written in terms of $t(q)$, the condition of absence of singularities in $V[4]$ takes the form

$$\begin{aligned}\frac{1}{4}\left(1+t^2(q)\right)^2 W_1(q, 0) &= \left(-t(q) + qt_q(q)\right)\left[3(-t(q) + qt_q(q)) + 6q^2 t_{qq}(q)\right. \\ &\quad \left.+ 2q^3 t_{qqq}(q)\right] - 3q^4 \left(\frac{d}{dq}(-t(q) + qt_q(q))\right)^2,\end{aligned}\tag{12}$$

in this expression, t_q is shorthand for dt/dq .

Now it is evident from (12) that if $t(q)$ satisfies

$$-t(q) + qt_q(q) = \beta,\tag{13}$$

the equation (12) becomes an identity and the condition (7) is satisfied provided that

$$W_1(q, 0) = \frac{12\beta^2}{(1 + t^2(q))^2}. \quad (14)$$

Integrating (13) we get

$$t(q) = \alpha q - \beta, \quad (15)$$

according to equation (9)

$$\delta(q) = \arctan(\alpha q - \beta), \quad (16)$$

in these expressions α and β are free parameters.

Once $\delta(q)$ is known as an explicit function of q , the functions γ_0 , γ_1 and γ_2 are obtained from its first, second and third derivatives

$$\begin{aligned} \gamma_0 &= \frac{d\delta(q)}{dq} = \frac{\alpha}{1 + (\alpha q - \beta)^2}, \\ \gamma_1 &= \frac{d^2\delta(q)}{dq^2} = -\frac{2\alpha^2(\alpha q - \beta)}{(1 + (\alpha q - \beta)^2)^2}, \\ \gamma_2 &= \frac{d^3\delta(q)}{dq^3} = -\frac{2\alpha^3(1 - 3(\alpha q - \beta)^2)}{(1 + (\alpha q - \beta)^2)^3}. \end{aligned} \quad (17)$$

With the help of these expressions and (5) we get for $W_1(q, r)$

$$\begin{aligned} W_1(q, r) &= \frac{12\beta^2}{(1 + (\alpha q - \beta)^2)^2} + \frac{24\beta\alpha q}{(1 + (\alpha q - \beta)^2)^2}(\cos 2qr - 1) \\ &+ \frac{12\alpha q((\alpha q)^2 + \beta^2 - 1)}{(1 + (\alpha q - \beta)^2)^2} \sin 2qr \\ &+ 16 \left[(qr)^4 + \frac{4\alpha q}{(1 + (\alpha q - \beta)^2)}(qr)^3 + \frac{6(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2}(qr)^2 \right. \\ &+ \left. \frac{3(\alpha q)^3}{(1 + (\alpha q - \beta)^2)^2}(qr) \right] - 12 \left[(qr)^2 + \frac{2\alpha q}{(1 + (\alpha q - \beta)^2)}(qr) \right] \\ &+ 24 \left[(qr)^2 + \frac{2\alpha q(1 - \beta(\alpha q - \beta))}{(1 + (\alpha q - \beta)^2)^2}(qr) \right] \cos 2(qr + \delta(q)) \\ &+ \left[16 \left((qr)^3 + \frac{3\alpha q}{(1 + (\alpha q - \beta)^2)}(qr)^2 + \frac{3(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2}(qr) \right) \right. \\ &- \left. 12(qr) \right] \sin 2(qr + \delta(q)) + 3 \left[\frac{1 - 6(\alpha q - \beta)^2 + (\alpha q - \beta)^4}{(1 + (\alpha q - \beta)^2)^2} \right. \\ &\times \left. \sin^2 2qr + \frac{4(\alpha q - \beta)(1 - (\alpha q - \beta)^2)}{(1 + (\alpha q - \beta)^2)^2} \sin 2qr \cos 2qr \right]. \end{aligned} \quad (18)$$

From this expression, we verify that the Wronskian $W_1(q, r)$, for $r = 0$, behaves as

$$W_1(q, 0) = \frac{12\beta^2}{(1 + (\alpha q - \beta)^2)^2}. \quad (19)$$

The asymptotic behaviour of $W_1(q, r)$ for large values of r is determined by the term $(q\gamma)^4$ which grows as r^4 and is the dominant term in the right hand side of eq.(5), with this result and equation (4), we get

$$\lim_{r \rightarrow \infty} V[4](r) \approx 8q \frac{\sin 2(qr + \delta(q))}{r} + O(r^{-2}) \quad (20)$$

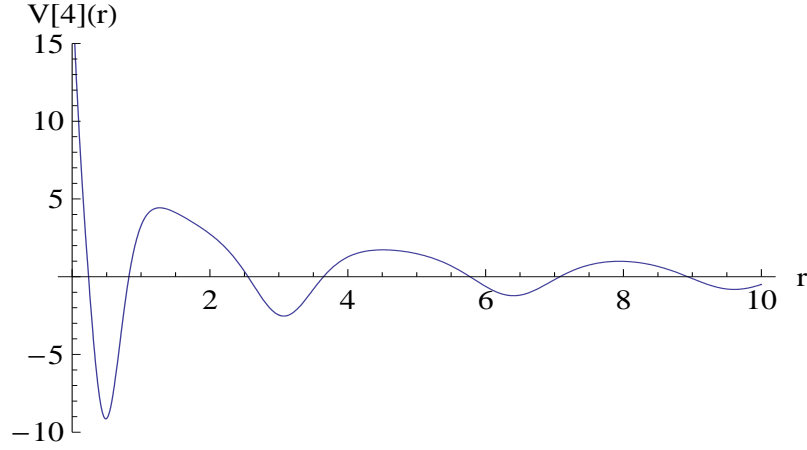


Figure 1. This graph shows the potential $V[4](r)$ for the values of the parameters $\alpha = 1$, $\beta = 3$ fixed at $q = 1$

Then, $V[4]$ is a potential of von Neumann-Wigner type [35]. The figure 1 shows the behaviour of $V[4]$ as function of r .

3. The Jost eigenfunctions of $H[4]$

The two linearly independent unnormalized Jost eigenfunctions of $H[4]$, that belong to the energy eigenvalues $E = k^2$ and behave as outgoing and incoming waves for large values of r are given in eq.(1). Notice that all terms in the last column of the Wronskian $W_2(\phi, \partial_q \phi, \dots, \partial_q^3 \phi, e^{\pm ikr})$ are proportional to $e^{\pm ikr}$.

Hence, the Jost solutions of eq.(1) take the form

$$f^\pm(k, r) = \frac{1}{W_1(q, r)} w^\pm(k, r) e^{\pm ikr}, \quad (21)$$

where the function $w^\pm(k, r)$ is the reduced Wronskian defined as

$$w^\pm(k, r) e^{\pm ikr} = W_2(\phi, \partial_q \phi, \dots, \partial_q^3 \phi, e^{\pm ikr}). \quad (22)$$

From this definition, it follows that $w^\pm(k, r)$ is a complex function of its arguments

$$w^\pm(k, r) = u(k, r) \pm iv(k, r). \quad (23)$$

A straightforward computation of the Wronskian $W_2(\phi, \partial_q \phi, \dots, \partial_q^3 \phi, e^{\pm ikr})$ allowed us to find the following explicit expressions for the functions $u(k, r)$ and $v(k, r)$

$$\begin{aligned} u(k, r) = & 12(k^4 + 6q^2 k^2 + q^4) \frac{\beta(\beta - 2\alpha q)}{(1 + (\alpha q - \beta)^2)^2} + 24(k^4 - 4q^2 k^2 - q^4) \\ & \times \frac{\beta \alpha q}{(1 + (\alpha q - \beta)^2)^2} \cos 2qr + 16(k^2 - q^2)^2 \left[(qr)^4 \right. \\ & + \frac{4\alpha q}{(1 + (\alpha q - \beta)^2)} (qr)^3 + \frac{6(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2} (qr)^2 \\ & \left. + \frac{3(\alpha q)^3}{(1 + (\alpha q - \beta)^2)^2} (qr) \right] + 24 \left[(k^4 - 4q^2 k^2 - q^4) \right. \end{aligned}$$

$$\begin{aligned}
& \times \left((qr)^2 + \frac{2\alpha q}{(1 + (\alpha q - \beta)^2)}(qr) \right) - 2(k^4 - q^4) \\
& \times \frac{(\alpha q)^2(\alpha q - \beta)}{(1 + (\alpha q - \beta)^2)^2}(qr) \Big] \cos 2(qr + \delta(q)) + \left[16(k^4 - q^4) \right. \\
& \times \left((qr)^3 + \frac{3\alpha q}{(1 + (\alpha q - \beta)^2)}(qr)^2 + \frac{3(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2}(qr) \right) \\
& \left. - 12(k^4 - 4q^2k^2 - q^4)(qr) \right] \sin 2(qr + \delta(q)) \\
& + \left[24(k^4 - q^4) \frac{(\alpha q)^3}{(1 + (\alpha q - \beta)^2)^2} - 12(k^4 - 4q^2k^2 - q^4) \right. \\
& \times \frac{\alpha q(1 + (\alpha q)^2 - \beta^2)}{(1 + (\alpha q - \beta)^2)^2} \Big] \sin 2qr \\
& + 3(k^4 + 6q^2k^2 + q^4) \left[\frac{1 - 6(\alpha q - \beta)^2 + (\alpha q - \beta)^4}{(1 + (\alpha q - \beta)^2)^2} \sin^2 2qr \right. \\
& \left. + \frac{2(\alpha q - \beta)(1 - (\alpha q - \beta)^2)}{(1 + (\alpha q - \beta)^2)^2} \right] \sin 2qr \cos 2qr
\end{aligned} \tag{24}$$

and

$$\begin{aligned}
v(k, r) = & 24qk \frac{\beta}{(1 + (\alpha q - \beta)^2)^2} \left[(\beta^2 - 4\beta\alpha q - 1)(k^2 + q^2) \right. \\
& \left. + (\alpha q)^2(q^2 + 5k^2) \right] + \left[24qk(k^2 + q^2) \right. \\
& \times \frac{\beta(\beta^2 - 4\beta\alpha q + (\alpha q)^2 + 1)}{(1 + (\alpha q - \beta)^2)^2} + 96qk^3 \frac{\beta(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2} \Big] \\
& \times \cos 2qr + 64qk(k^2 - q^2) \left[(qr)^3 + \frac{3\alpha q}{1 + (\alpha q - \beta)^2}(qr)^2 \right. \\
& \left. + \frac{3(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2}(qr) \right] - 24qk(k^2 + q^2)(qr) + \left[32qk(k^2 - q^2) \right. \\
& \times \left((qr)^3 + \frac{3\alpha q}{1 + (\alpha q - \beta)^2}(qr)^2 + \frac{3(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2}(qr) \right) \\
& \left. + 24qk(k^2 + q^2)(qr) \right] \cos 2(qr + \delta(q)) \\
& + \left[96q^3k \left((qr)^2 + \frac{2\alpha q}{1 + (\alpha q - \beta)^2}(qr) \right) \right. \\
& \left. + 96qk(k^2 - q^2) \frac{(\alpha q)^2(\alpha q - \beta)}{(1 + (\alpha q - \beta)^2)^2}(qr) \right] \sin 2(qr + \delta(q)) \\
& + \left[12qk(k^2 + q^2) \frac{(\alpha q - \beta)^4 + 4\beta\alpha q - 1}{(1 + (\alpha q - \beta)^2)^2} \right. \\
& \left. - 48qk(k^2 - q^2) \frac{(\alpha q)^2}{(1 + (\alpha q - \beta)^2)^2} \right] \sin 2qr \\
& + 12qk(k^2 + q^2) \left[\frac{1 - 6(\alpha q - \beta)^2 + (\alpha q - \beta)^4}{(1 + (\alpha q - \beta)^2)^2} \sin 2qr \cos 2qr \right. \\
& \left. - \frac{4(\alpha q - \beta)(1 - (\alpha q - \beta)^2)}{(1 + (\alpha q - \beta)^2)^2} \sin^2 2qr \right].
\end{aligned} \tag{25}$$

For large values of r , the asymptotic behaviour of $w^\pm(k, r)$ is dominated by the highest power of r . From eqs.(24) and (25) we get

$$w^\pm(k, r) \approx 16(k^2 - q^2)^2 (qr)^4 \left\{ 1 + O(r^{-1}) \right\} e^{\pm ikr} \quad (26)$$

and from eq.(18) we get

$$W_1(q, r) \approx 16(qr)^4 [1 + O(r^{-1})], \quad (27)$$

hence, for large values of r

$$f^\pm(k, r) \approx (k^2 - q^2)^2 [1 + O(r^{-1})] e^{\pm ikr}. \quad (28)$$

The factor $(k^2 - q^2)^2$ is the flux of probability current at infinity of the unnormalized Jost solutions.

Therefore, the Jost solutions of $H[4]$ normalized to unit probability flux at infinity are

$$F^\pm(k, r) = \frac{f^\pm(k, r)}{(k^2 - q^2)^2} = \frac{1}{(k^2 - q^2)^2} \frac{w^\pm(k, r)}{W_1(q, r)} e^{\pm ikr}, \quad k^2 \neq q^2. \quad (29)$$

4. Exceptional points in the spectrum of $H[4]$

Each pair of linearly independent Jost solutions belongs to a point $E_k = k^2$, with $k^2 \neq q^2$, in the spectrum of $H[4]$. At the point $E_q = q^2$, the two unnormalized Jost solutions coalesce to give rise to a Jordan chain of two generalized bound state eigenfunctions of $H[4]$ [36].

The Wronskian of the unnormalized Jost solutions is readily computed from (21) and (23)

$$W(f^+(k, r), f^-(k, r)) = -2ik(k + q)^4(k - q)^4, \quad (30)$$

At the points $k = \pm q$, the Wronskian of the two unnormalized Jost solutions of $H[4]$ vanishes, then the two unnormalized Jost solutions are no longer linearly independent, coalesce in one bound state eigenfunction embedded in the continuum. This property identifies the points $k = \pm q$ as exceptional points in the spectrum of the Hamiltonian $H[4]$.

5. Jordan cycle of generalized eigenfunctions and Jordan block representation of $H[4]$

The normalized Jost solutions of $H[4]$, as functions of the wave number k , may be written as the sum of a singular and a regular part [36]

$$\begin{aligned} F^\pm(k, r) = & \frac{\psi_B(q, r)e^{\mp i\delta(q)}}{(k - q)^2} + \frac{\chi_B^\pm(q, r)e^{\mp i\delta(q)}}{(k - q)} + \frac{\psi_B(-q, r)e^{\pm i\delta(q)}}{(k + q)^2} \\ & + \frac{\chi_B^\pm(-q, r)e^{\pm i\delta(q)}}{(k + q)} + h^\pm(k, r), \end{aligned} \quad (31)$$

explicit expressions for the functions $\psi_B(q, r)$ and $\chi_B^\pm(q, r)$ as functions of r , are

$$\psi_B(q, r) = \frac{24q^2}{W_1(q, r)} \{-2q^2\gamma^2 \cos \theta + [q\gamma + q^2\gamma_1] \sin \theta + \sin^2 \theta \cos \theta\} \quad (32)$$

and

$$\chi_B^\pm(q, r) = \chi_B(q, r) \mp i\gamma_0\psi_B(q, r), \quad (33)$$

where

$$\begin{aligned} \chi_B(q, r) = \frac{8q}{W_1(q, r)} & [-2q^3\gamma^3 \sin \theta - 3q^2\gamma^2 \cos \theta + 3q\gamma \sin^3 \theta \\ & - \gamma_2 q^3 \sin \theta + 3\gamma_1 \gamma q^3 \cos \theta + 3 \sin^2 \theta \cos \theta]. \end{aligned} \quad (34)$$

The functions γ_0, γ_1 and γ_2 are given in equations (17)

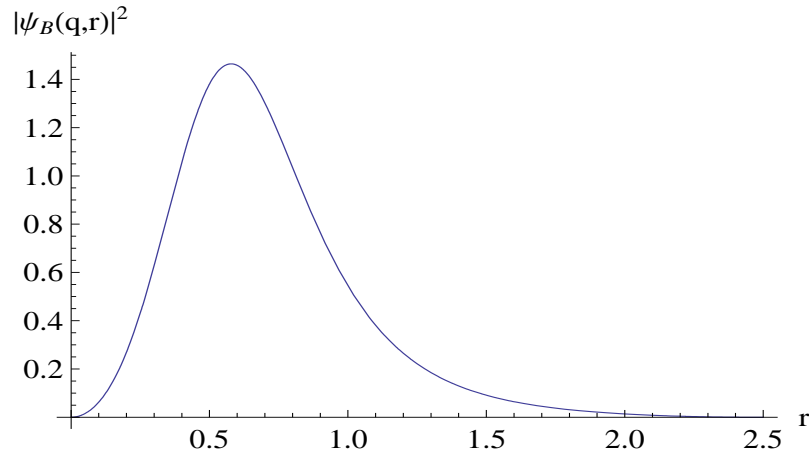


Figure 2. This graph shows the bound state eigenfunction $\psi_B(q, r)$ as function of r computed for $q = 1$ and the values of the parameters $\alpha = 1$ and $\beta = 3$

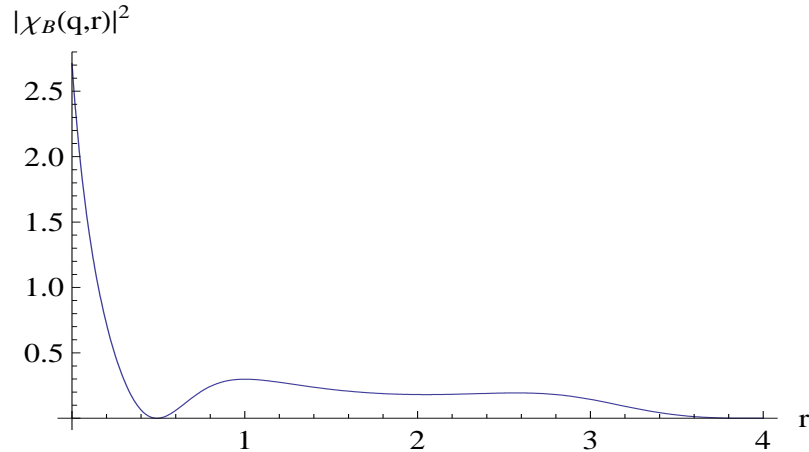


Figure 3. The generalized eigenfunction $\chi_B(q, r)$, as a function of r computed for $q = 1$ and the values of the parameters $\alpha = 1$ and $\beta = 3$.

In the figures 2 and 3 we show the graphical representation of the generalized eigenfunctions $\psi_B(q, r)$ and $\chi_B(q, r)$ as function of r .

The functions $\psi_B(q, r)$ and $\chi_B(q, r)$ are the elements of a Jordan cycle of generalized eigenfunctions of the Hamiltonian that belong to the same point $k = q$ in its spectrum, and satisfy the set of coupled equations

$$H[4]\psi_B(q, r) = q^2\psi_B(q, r) \quad (35)$$

and

$$H[4]\chi_B(q, r) = q^2\chi_B(q, r) + 2q\psi_B(q, r). \quad (36)$$

This Jordan cycle of generalized eigenfunctions is associated with a Jordan block representation of the Hamiltonian $H[4]$. This property is made evident if the equations (35) and (36) are written in matrix form

$$\left[H[4]\mathbf{1}_{2 \times 2} \right] \Psi_B(q, r) = \mathcal{H}_B(q)\Psi_B(q, r), \quad (37)$$

where

$$\Psi_B(q, r) = \begin{pmatrix} \psi_B(q, r) \\ \chi_B(q, r) \end{pmatrix} \quad (38)$$

and

$$\mathcal{H}_B(q) = \begin{pmatrix} q^2 & 0 \\ 2q & q^2 \end{pmatrix}. \quad (39)$$

From (37), it is evident that the matrix $\mathcal{H}_B(q)$ is a matrix representation of the Hamiltonian $H[4]$ in the two dimensional functional space spanned by the generalized eigenfunctions $\{\psi_B(q, r), \chi_B(q, r)\}$.

This space is a subset of the rigged Hilbert space of continuous, complex functions of the variables (q, r) with continuous first and second derivatives with respect to r , with r in the semi-infinite straight line $0 \leq r < \infty$. Therefore, the two dimensional subspace of functions spanned by the generalized eigenfunctions $\{\psi_B(q, r), \chi_B(q, r)\}$ is in the domain of $H[4]$.

The matrix $\mathcal{H}_B(q)$ is a Jordan block of 2×2 [41] and can not be brought to diagonal form by means of a similarity transformation with a unitary matrix [42].

The real non-symmetric matrix $\mathcal{H}_B(q)$ is η -pseudo-Hermitian [14]

$$\mathcal{H}_B^\dagger(q) = \eta^{-1}\mathcal{H}_B(q)\eta \quad (40)$$

with

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (41)$$

Hence, also the Hamiltonian operator $H[4]$ when acting on the functional space spanned by the generalized eigenfunctions $\{\psi_B(q, r), \chi_B(q, r)\}$ is also η -pseudo-Hermitian.

6. Scattering solutions and the scattering matrix

In this section it will be shown that the regular scattering solution $\psi_s(k, r)$ vanishes at $k = \pm q$, and the irregular scattering solution $\psi_{is}(k, r)$ has a double pole at the exceptional points $k = \pm q$.

The regular scattering solution of $H[4]$ is given by

$$\psi_s(k, r) = \frac{i}{2} \left[F^-(k, r) - S(k) F^+(k, r) \right], \quad k \neq q, \quad (42)$$

where the scattering matrix $S(k)$ is

$$S(k) = \frac{f^-(k, 0)}{f^+(k, 0)} = \frac{w^-(k, 0)}{w^+(k, 0)} = \frac{u(k, 0) - iv(k, 0)}{u(k, 0) + iv(k, 0)}. \quad (43)$$

From the Jost solutions eq.(29), we get

$$\begin{aligned} \psi_s(k, r) = \frac{i}{2} \frac{1}{(k^2 - q^2)^2} \frac{1}{W_1(q, r)} \frac{1}{w^+(k, 0)} \left[w^+(k, 0) w^-(k, r) e^{-ikr} \right. \\ \left. - w^-(k, 0) w^+(k, r) e^{ikr} \right]. \end{aligned} \quad (44)$$

a graphical representation of the regular scattering solution is shown in figure 4.

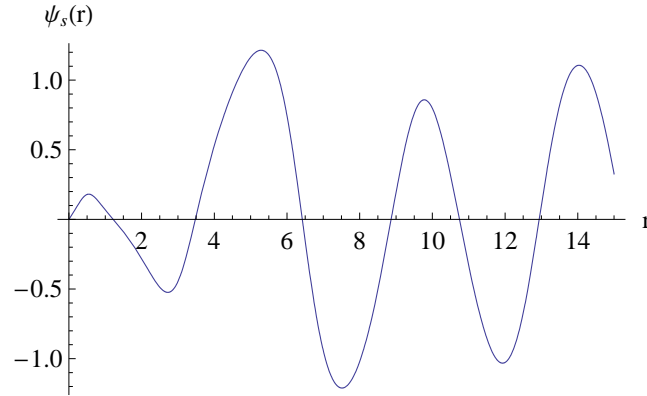


Figure 4. The graph shows the regular scattering solution as function of r for the values of the parameters in $V[4]$ and $k = 1.5$.

In order to make explicit the properties of the scattering solutions as functions of k , for k close to the exceptional points at $k = \pm q$, we will write the reduced Wronskian $w^\pm(k, r)$ as an expansion in powers of $(k - q)$

$$w^\pm(k, r) = u(k, r) \pm iv(k, r) = e^{\mp i\theta(q)} \sum_{\ell=0}^4 w_\ell^\pm(q, r) (k - q)^\ell, \quad (45)$$

explicit expressions for the coefficients $w_\ell^\pm(q, r)$, as functions of their arguments are given in the appendix A.

Then, the terms in the right hand side of eq.(44) may also be expressed as an expansion in powers of $(k - q)$

$$\begin{aligned} \psi_s(k, r) = & \frac{i}{2} \frac{1}{(k^2 - q^2)^2} \frac{1}{W_1(q, r)} \frac{1}{w^+(k, 0)} \sum_{\ell=0}^4 \sum_{m=0}^4 \left[w_\ell^+(q, 0) w_m^-(q, r) \right. \\ & \left. \times e^{-i(k-q)r} - w_\ell^-(q, 0) w_m^+(q, r) e^{i(k-q)r} \right] (k - q)^{(\ell+m)}, \end{aligned} \quad (46)$$

In appendix B, we have shown that the first three terms in the summation in the right hand side of eq.(46) vanish.

It follows that

$$\begin{aligned} \psi_s(k, r) = & \frac{i}{2} \frac{(k - q)}{(k + q)^2 W_1(q, r)} \frac{1}{w^+(k, 0)} \sum_{\ell=3}^8 \sum_{m=0}^{\ell} \left[w_{\ell-m}^+(q, 0) w_m^-(q, r) e^{-i(k-q)r} \right. \\ & \left. - w_{\ell-m}^-(q, 0) w_m^+(q, r) e^{i(k-q)r} \right] (k - q)^{\ell-3} + O(k - q) \end{aligned} \quad (47)$$

with the restriction $w_\ell^\pm(q, r) = 0$, for $\ell \geq 5$.

Therefore, at the excepcional point, the regular scattering solution vanishes,

$$\psi_s(q, r) = 0. \quad (48)$$

The energy eigenfunctions at the exceptional point are the generalized bound state eigenfunctions $\psi_B(q, r)$ and $\chi_B(q, r)$.

Let us now turn our attention to the irregular scattering solution,

$$\psi_{is}(k, r) = \frac{1}{2} \left[F^-(k, r) + S(k) F^+(k, r) \right], \quad k \neq q. \quad (49)$$

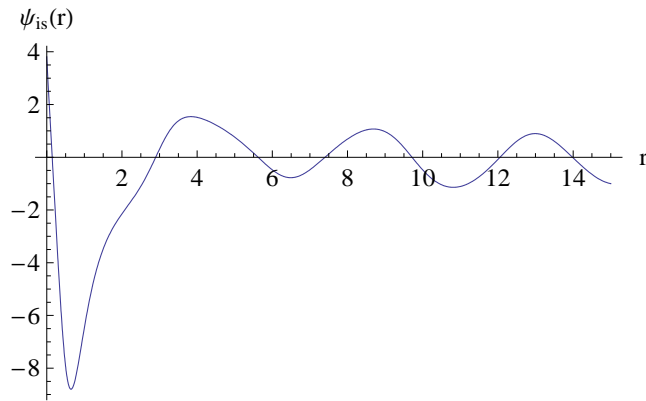


Figure 5. The graph shows the irregular scattering solution as function of r for the values of the parameters in $V[4]$ and $k = 1.5$.

In this case, we have

$$\begin{aligned} \psi_{is}(k, r) = & \frac{1}{2} \frac{1}{(k^2 - q^2)^2} \frac{1}{W_1(q, r)} \frac{1}{w^+(k, 0)} \left[w^+(k, 0) w^-(k, r) e^{-ikr} \right. \\ & \left. + w^-(k, 0) w^+(k, r) e^{ikr} \right]. \end{aligned} \quad (50)$$

Notice that, when the expression (45) is substituted for $w^\pm(k, r)$ in (50) we obtain an expansion of the right hand side of (50) in powers of $(k - q)$ similar to (46), but now

the terms proportional to $w_0^\pm(q, 0)w_0^\pm(q, r)$, $w_0^\pm(q, 0)w_1^\pm(q, r)$ and $w_1^\pm(q, 0)w_1^\pm(q, r)$ are added. Hence, the singular terms in the expressions (31) for the Jost solutions add instead of cancelling which makes $\psi_{is}(k, r)$ singular at the exceptional points. Hence, the irregular scattering solution $\psi_{is}(k, r)$, as a function of k , has a pole of second order at the exceptional points.

Substitution of the expressions (45) for $w^\pm(k, r)$ in (50) and multiplying the result times $(k - q)^2$ and taking the limit $k \rightarrow q$ gives,

$$\lim_{k \rightarrow q} (k - q)^2 \psi_{is}(k, r) = \psi_B(q, r). \quad (51)$$

Therefore, the bound state eigenfunction $\psi_B(q, r)$ is the residue of second order of the irregular scattering solution $\psi_{is}(k, r)$ at $k = q$.

From equations (24) and (25) for the funciones $u(k, r)$ and $v(k, r)$ evaluated in $r = 0$ we get

$$u(k, 0) = \frac{12\beta}{(1 + (\alpha q - \beta)^2)^2} [\beta k^4 + (6\beta - 20\alpha q)k^2 q^2 + (\beta - 4\alpha q)q^4] \quad (52)$$

and

$$v(k, 0) = \frac{48\beta q k}{(1 + (\alpha q - \beta)^2)^2} \times [(\beta^2 - 4\beta\alpha q + 5\alpha^2 q^2)k^2 + (\beta^2 - 4\beta\alpha q + \alpha^2 q^2)q^2], \quad (53)$$

substitution of the equations (52) and (53) in (43) allows to write the following expression for the scattering matrix $S(k)$

$$S(k) = \exp 2i\Delta(k) \quad (54)$$

where the phase shift $\Delta(k)$ is

$$\Delta(k) = -\arctan \frac{4qk [(\beta^2 - 4\beta\alpha q + 5\alpha^2 q^2)k^2 + (\beta^2 - 4\beta\alpha q + \alpha^2 q^2)q^2]}{\beta k^4 + (6\beta - 20\alpha q)q^2 k^2 + (\beta - 4\alpha q)q^4}, \quad (55)$$

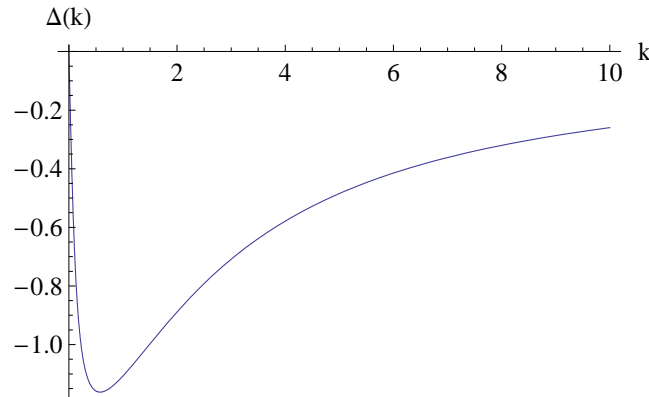


Figure 6. The phase shift $\Delta(k)$ as function of the wave number k for $q = 1$ and the values of the parameters $\alpha = 1$ and $\beta = 3$.

the figure 6 shows the graphical representation of the phase shift.

The scattering matrix is a regular analytical function of the wave number k , including the exceptional points, it does not have any singularity or pole.

7. Unitarity time evolution of the regular scattering eigenfunctions of $H[4]$

The regular time dependent wave functions built as a linear combination of the regular scattering eigenfunctions of $H[4]$ is

$$\Psi_r(r, t) = \int C(k) e^{-i \frac{k^2(t-t_0)}{\hbar}} \psi_s(k, r) dk. \quad (56)$$

This is a solution of the time dependent Schrödinger equation

$$i\hbar \frac{\partial \Psi_r(r, t)}{\partial t} = H[4] \Psi_r(r, t), \quad (57)$$

as it may be verified by substitution of (56) in (57).

The coefficient $C(k)$ occurring in the right hand side of eq.(56) is a quadratically integrable function of k .

Since the regular scattering eigenfunctions vanishes at the exceptional point

$$\psi_s(q, r) = 0, \quad (58)$$

it does not contribute to the integral in the right hand side of eq.(56).

Therefore, the presence of an exceptional point in the spectrum of $H[4]$ does not alter the unitarity evolution of the regular time dependent wave function.

8. Pseudounitariness time evolution of the irregular scattering eigenfunctions

The two generalized eigenfunctions $\psi_B(q, r)$ and $\chi_B(q, r)$ belong to the same spectral point, $E_q = q^2$, in consequence, they evolve in time together. Hence, it should be convenient to introduce a matrix notation to deal with the two together,

$$\Psi(r, t) = C(q, t) \Psi_B(q, r), \quad (59)$$

where $\Psi_B(q, r)$ is the two component vector of the doublet

$$\Psi_B(q, r) = \begin{pmatrix} \psi_B(q, r) \\ \chi_B(q, r) \end{pmatrix} \quad (60)$$

and $C(q, t)$ is the 2×2 matrix of time dependent coefficient of the wave functions $\psi_B(q, r)$ and $\chi_B(q, r)$.

Substitution of $\Psi(r, t)$ in the time dependent Schrödinger equation gives the following set of coupled equations written in matrix form

$$i\hbar \frac{\partial C(q, t)}{\partial t} \Psi_B(q, r) = C(q, t) H[4] \mathbf{1}_{2 \times 2} \Psi_B(q, r) = C(q, t) \mathcal{H}_B(q) \Psi_B(q, r), \quad (61)$$

making abstraction of $\Psi_B(q, r)$, we obtain

$$i\hbar \frac{\partial C(q, t)}{\partial t} = C(q, t) \mathcal{H}_B(q), \quad (62)$$

where

$$\mathcal{H}_B(q) = \begin{pmatrix} q^2 & 0 \\ 2q & q^2 \end{pmatrix}. \quad (63)$$

The matrix $\mathcal{H}_B(q)$ is a representation of the Hamiltonian $H[4]$ in the two dimensional functional space spanned by the generalized eigenfunctions $\{\psi_B(q, r), \chi_B(q, r)\}$.

Integrating eq.(62) we get

$$C(q, t) = e^{-i\frac{1}{\hbar}\mathcal{H}(q)t}, \quad (64)$$

writing $\mathcal{H}_B(q)$ in explicit form in (64), and computing the exponential, we obtain

$$C(q, t) = e^{-i\frac{q^2 t}{\hbar}} \begin{pmatrix} 1 & 0 \\ -i\frac{2qt}{\hbar} & 1 \end{pmatrix}. \quad (65)$$

Substitution of the expressions (65) for $C(q, t)$ and (60) in (59) gives

$$\psi_B(r, t) = e^{-i\frac{q^2 t}{\hbar}} \psi_B(q, r) \quad (66)$$

and

$$\chi_B(r, t) = e^{-i\frac{q^2 t}{\hbar}} \chi_B(q, r) - i\frac{2q}{\hbar} t e^{-i\frac{q^2 t}{\hbar}} \psi_B(q, r). \quad (67)$$

9. Summary and conclusions

An example of exceptional points in the continuous spectrum of a real, pseudo-Hermitian Hamiltonian $H[4]$ of von Neumann-Wigner type is presented and discussed. The Hamiltonian $H[4]$ and the free particle Hamiltonian H_0 are isospectral. In the general case, to each point in this continuous spectrum, correspond two linearly independent Jost solutions which behave at infinity as incoming and outgoing waves. Clearly in both cases, this continuous spectrum is doubly degenerate. However, here we have shown that in the continuous spectrum of $H[4]$ there are two exceptional points at wave numbers $k = \pm q$, these exceptional points are associated with a double pole in the normalization factor of the Jost eigenfunctions normalized to unit flux at infinity. At the exceptional points, the two unnormalized Jost eigenfunctions are no longer linearly independent and coalesce to give rise to Jordan cycles of length two of generalized quadratically integrable bound states eigenfunctions embedded in the continuum and a Jordan block representation of the Hamiltonian $H[4]$. The regular scattering eigenfunction vanishes at the exceptional point and the irregular scattering eigenfunction has a double pole with a second order residue equal to the bound state eigenfunction in the continuum and a Jordan block representation of the Hamiltonian. In consequence, the time evolution of the regular scattering eigenfunction is unitary, while the time evolution of the irregular scattering eigenfunction is pseudounitary. The scattering matrix $S(k)$ is a regular function of k at the exceptional points, that is, the Jordan cycle of generalized bound states eigenfunctions of $H[4]$ in the continuum is not associated with a pole of the scattering matrix.

9.1. Acknowledgments

This work was partially supported by CONACyT México under Contract No. 132059 and by DGAPA-UNAM Contract No. PAPIIT:IN113712.

Appendix A.

The dependence of $w^\pm(k, r)$ on k is readily made evident by expanding the reduced Wronskian defined in eq.(45) in powers of $(k - q)$

$$w^\pm(k, r) = e^{\mp i\theta} \sum_{\ell=0}^4 w_\ell^\pm(q, r)(k - q)^\ell, \quad (\text{A.1})$$

the coefficients of the powers of $(k - q)$ in this expansion are

$$w_0^\pm(q, r) = 4q^2 W_1(q, r) \psi_B(q, r) \quad (\text{A.2})$$

$$w_1^\pm(q, r) = 4q W_1(q, r) \left[\psi_B(q, r) + q \chi_B(q, r) \mp i q \gamma \psi_B(q, r) \right] \quad (\text{A.3})$$

$$\begin{aligned} w_2^\pm(q, r) = & -2W_1(q, r) \psi_B(q, r) \cos^2 \theta + 6q W_1(q, r) \chi_B(q, r) \\ & + 16q^2 \left\{ 4q^4 \gamma^4 - 3q^2 \gamma^2 - 3q^4 \gamma_1^2 + 2q^4 \gamma \gamma_2 + 3 \sin^2 \theta \cos^2 \theta \right\} \cos \theta \\ & \mp i 2W_1(q, r) \left\{ (2q\gamma + q^2 \gamma_1) \psi_B(q, r) + 2q^2 \gamma \chi_B(q, r) \right\} \end{aligned} \quad (\text{A.4})$$

$$\begin{aligned} \omega_3^\pm(q, r) = & 2W_1(q, r) \chi_B(q, r) - \frac{4}{q} W_1(q, r) \psi_B(q, r) \\ & + 8q \left((8q^4 \gamma^4 - 18q^2 \gamma^2 + 3q^3 \gamma_1 \gamma - 6q^4 \gamma_1^2) \cos \theta \right. \\ & + (8q^3 \gamma^3 + 3q\gamma - 12q^2 \gamma_1 + 2q^3 \gamma_2^2) \sin^3 \theta \\ & \left. - 12(2q^2 \gamma^2 + q^3 \gamma_1 \gamma - 1) \sin^2 \theta \cos \theta \right) \\ & \mp i \left[4q\gamma W_1(q, r) \chi_B(q, r) + 2(\gamma + \gamma_1) W_1(q, r) \psi_B(q, r) \right. \\ & - 8q \left((12q^3 \gamma^3 - 6q^2 \gamma_1 - 3q^3 \gamma_2) \sin^2 \theta \cos \theta \right. \\ & \left. \left. + 12(q^2 \gamma^2 + 3q^3 \gamma_1 \gamma) \cos^2 \theta \sin \theta - 12q^3 \gamma^3 \cos \theta \right) \right] \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} \omega_4^\pm(q, r) = & \left[-\frac{1}{q^2} W_1(q, r) \psi_B(q, r) \cos \theta + 8(4q^3 \gamma^3 + q^3 \gamma_2) \sin \theta \cos \theta \right. \\ & + 16q^4 \gamma^4 - 36q^2 \gamma^2 + 12q^2 \gamma_1 (2q\gamma - q^2 \gamma_1) + 36 \sin^2 \theta \cos^2 \theta \\ & \left. - 48q^3 \gamma_1 \gamma \sin^2 \theta \right] \cos \theta \pm i \left[-\frac{1}{q^2} W_1(q, r) \psi_B(q, r) \cos \theta \right. \\ & + 8(4q^3 \gamma^3 + q^3 \gamma_2) \sin \theta \cos \theta + 16q^4 \gamma^4 - 36q^2 \gamma^2 \\ & + 12q^2 \gamma_1 (2q\gamma - q^2 \gamma_1) + 36 \sin^2 \theta \cos^2 \theta - 48q^3 \gamma_1 \gamma \sin^2 \theta \left. \right] \\ & \times \sin \theta \end{aligned} \quad (\text{A.6})$$

An expansion of the reduced Wronskian, $w^\pm(k, r)$, about the point $k = -q$ is readily obtained from eq. (A.1). From time reversal invariance of the Hamiltonian plus boundary conditions we get

$$f^+(-k, r) = f^-(k, r) \quad (\text{A.7})$$

and

$$w^\pm(k, r) = w^\mp(-k, r). \quad (\text{A.8})$$

Hence, from (A.1)

$$w^\pm(k, r) = e^{\mp i\theta(q)} \sum_{\ell=0}^4 (-1)^\ell w_\ell^\pm(q, r) (k+q)^\ell, \quad (\text{A.9})$$

the coefficients $w_\ell^\pm(q, r)$ are explicitly given in eqs.(A.2 - A.6) as entire functions of q . Their domain of analiticity is trivially extended to negative values of the argument q just by changing q by $-q$ in (A.2 - A.6). Then it may be verified that

$$(-1)^\ell w_\ell^\pm(q, r) = w_\ell^\pm(-q, r). \quad (\text{A.10})$$

Therefore,

$$w^\pm(k, r) = e^{\pm i\theta(-q)} \sum_{\ell=0}^4 (-1)^\ell w_\ell^\pm(|q|, r) (k+q)^\ell, \quad k < 0. \quad (\text{A.11})$$

Appendix B.

In this appendix it will be shown that the regular scattering eigenfunction vanishes at the exceptional points $k = \pm q$.

The regular scattering solution, eq.(29), is

$$\begin{aligned} \psi_s(k, r) = \frac{i}{2} \frac{1}{(k^2 - q^2)^2} \frac{1}{W_1(q, r)} \frac{1}{w^+(k, 0)} \Big[& w^+(k, 0) w^-(k, r) e^{-ikr} \\ & - w^-(k, 0) w^+(k, r) e^{ikr} \Big]. \end{aligned} \quad (\text{B.1})$$

With the reduced Wronskian $w^\pm(k, r)$ as an expansion in powers of $(k-q)$, eq.(A.1), we get

$$\begin{aligned} \psi_s(k, r) = \frac{i}{2} \frac{1}{(k^2 - q^2)^2} \frac{1}{W_1(q, r)} \frac{1}{w^+(k, 0)} \sum_{\ell=0}^8 \sum_{m=0}^{\ell} \Big[& w_{\ell-m}^+(q, 0) w_m^-(q, r) \\ & \times e^{-i(k-q)r} - w_{\ell-m}^-(q, 0) w_m^+(q, r) e^{i(k-q)r} \Big] (k-q)^\ell, \end{aligned} \quad (\text{B.2})$$

with the restriction $w_\ell^\pm(q, r) = 0$ for $\ell \geq 5$.

Writing the first three terms in the summation, corresponding to $\ell = 0, 1$ and 2 , and using the expansion of $e^{-i(k-q)r}$, we get

$$\begin{aligned} \psi_s(k, r) = \frac{i}{2} \frac{1}{(k^2 - q^2)^2} \frac{1}{W_1(q, r)} \frac{1}{w^+(k, 0)} \{ & \zeta_0(q, r) \\ & + \zeta_1(q, r)(k-q) + \zeta_2(q, r)(k-q)^2 + O((k-q)^3) \\ & + \sum_{\ell=3}^8 \sum_{m=0}^{\ell} \Big[w_{\ell-m}^+(q, 0) w_m^-(q, r) e^{-i(k-q)r} \\ & - w_{\ell-m}^-(q, 0) w_m^+(q, r) e^{i(k-q)r} \Big] (k-q)^\ell \}, \end{aligned} \quad (\text{B.3})$$

where the functions $\zeta_0(q, r)$, $\zeta_1(q, r)$ y $\zeta_2(q, r)$ are given by

$$\zeta_0(q, r) = w_0^+(q, 0) w_0^-(q, r) - c.c. \quad (\text{B.4})$$

$$\zeta_1(q, r) = -i w_0^+(q, 0) w_0^-(q, r) r + w_1^+(q, 0) w_0^-(q, r)$$

$$+ w_0^+(q, 0)w_1^-(q, r) - c.c. \quad (B.5)$$

$$\begin{aligned} \zeta_2(q, r) = & -\frac{1}{2}w_0^+(q, 0)w_0^-(q, r)r^2 - iw_1^+(q, 0)w_0^-(q, r)r \\ & - iw_0^+(q, 0)w_1^-(q, r)r + w_2^+(q, 0)w_0^-(q, r)r \\ & + w_1^+(q, 0)w_1^-(q, r) + w_0^+(q, 0)w_2^-(q, r) - c.c. \end{aligned} \quad (B.6)$$

in these expressions, c.c. is shorthand for complex conjugate.

From the eq.(A.2), $w_0^+(q, r) = w_0^-(q, r) = w_0(q, r)$, where $w_0(q, r)$ is the real function

$$w_0(q, r) = 4q^2W_1(q, r)\psi_B(q, r) \quad (B.7)$$

then, from eq.(B.4),

$$\zeta_0(q, r) = 0. \quad (B.8)$$

From eq.(A.3) we note that the imaginary part of $w_1^\pm(q, r)$ is

$$w_1^\pm(q, r) = \mp 4W_1(q, r)q^2\gamma\psi_B(q, r), \quad (B.9)$$

hence, $\zeta_1(q, r)$ is written as

$$\begin{aligned} \zeta_1(q, r) = & -2iw_0(q, 0)w_0(q, r)r - 8iq^2\gamma_0W_1(q, 0)\psi_B(q, 0)w_0(q, r) \\ & + 8iq^2\gamma w_0(q, 0)W_1(q, r)\psi_B(q, r) \end{aligned} \quad (B.10)$$

and using (B.7) we get

$$\zeta_1(q, r) = 2iw_0(q, 0)w_0(q, r)(-r - \gamma_0 + \gamma). \quad (B.11)$$

But, from eq.(6), $\gamma = r + \gamma_0$, then

$$\zeta_1(q, r) = 0. \quad (B.12)$$

The function $\zeta_2(q, r)$ is computed from the expressions for $w_1^\pm(q, r)$ and $w_2^\pm(q, r)$, we get

$$\begin{aligned} \zeta_2(q, r) = & -8iqW_1(q, 0)[\psi_B(q, 0) + q\chi_B(q, 0)]w_0(q, r)r \\ & - 8iqW_1(q, r)[\psi_B(q, r) + q\chi_B(q, r)]w_0(q, 0)r \\ & + 32iq^2W_1(q, 0)[\psi_B(q, 0) + q\chi_B(q, 0)]W_1(q, r)q\gamma\psi_B(q, r) \\ & - 32iq^2W_1(q, r)[\psi_B(q, r) + q\chi_B(q, r)]W_1(q, 0)q\gamma_0\psi_B(q, 0) \\ & - 4iW_1(q, 0)[(2q\gamma_0 + q^2\gamma_1)\psi_B(q, 0) + 2q^2\gamma_0\chi_B(q, 0)]w_0(q, r) \\ & + 4iW_1(q, r)[(2q\gamma + q^2\gamma_1)\psi_B(q, r) + 2q^2\gamma\chi_B(q, r)]w_0(q, 0). \end{aligned} \quad (B.13)$$

Using eq.(B.7) for $w_0(q, r)$, we get

$$\begin{aligned} \zeta_2(q, r) = & 32iq^3W_1(q, r)W_1(q, 0)\{2\psi_B(q, r)\psi_B(q, 0) \\ & + q[\psi_B(q, r)\chi_B(q, 0) + \psi_B(q, 0)\chi_B(q, r)]\}(-r - \gamma_0 + \gamma). \end{aligned} \quad (B.14)$$

But, $\gamma = r + \gamma_0$, then

$$\zeta_2(q, r) = 0. \quad (B.15)$$

With these results, the regular scattering solution $\psi_s(k, r)$, as function of k , close to the exceptional point $k = q$, is

$$\begin{aligned} \psi_s(k, r) = & \frac{i}{2} \frac{1}{(k+q)^2} \frac{k-q}{W_1(q, r)} \frac{1}{w^+(k, 0)} \sum_{\ell=3}^8 \sum_{m=0}^{\ell} \left[w_{\ell-m}^+(q, 0) w_m^-(q, r) \right. \\ & \times e^{-i(k-q)r} - w_{\ell-m}^-(q, 0) w_m^+(q, r) e^{i(k-q)r} \left. \right] (k-q)^{\ell-3} \\ & + O((k-q)). \end{aligned} \quad (\text{B.16})$$

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